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THE JOHNS HOPKINS UNIVERSITY
Department of Civil Engineering
BALTIMORE, MARYLAND
March, 1954

HARMONIC WAVE SOLUTIONS OF THE NONLINEAR VORTICITY EQUATION FOR A ROTATING VISCOUS FLUID

By Shih-Kung Kao

Technical Report No. 3

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Sponsored by

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The research reported in this document was supported by The Office of Naval Research under Project NR 082-104, Contract N-onr-248(31).

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ABSTRACT

With the assumption of non-divergent horizontal flow, harmonic wave solutions of the nonlinear hydrodynamic equations for a rotating, viscous fluid are obtained both for the plane and for the sphere. These solutions yield waves which may be damped, amplified, or remain unchanged, depending on the longitudinal and latitudinal extents of the waves, and also on the vertical profile of its stream function. For plane flow, the vertical profile of the velocity components is shown to be a sinusoidal function, or a hyperbolic sine and cosine of z, proportional to or independent of z. For spherical flow, the vertical profile is shown to be proportional to a Bessel function of order $n + \frac{1}{2}$ and n, a positive integer, is the order of an associated Legendre function. Harmonic waves are amplified, neutral or damped, depending respectively on whether the argument of the Bessel function is imaginary, zero or real. No stationary wave is possible in spherical viscous flow, although quasi-stationary waves, which may be defined as waves with zero wave-velocity but changing wave-amplitude, may occur under certain circumstances.

HARMONIC WAVE SOLUTIONS OF THE NONLINEAR VORTICITY EQUATION FOR A ROTATING VISCOUS FLUID

1. Introduction

The linearized vorticity equation for a rotating invisid fluid was first studied by Rossby (1939), who obtained a solution for the motion of sinusoidal waves of infinite lateral extent on an earth whose variation of Coriolis paramter with latitude is constant. Later Haurwitz (1940) extended Rossby's method and obtained solutions for waves of finite lateral extent on a horizontal plane (1940a) and on a sphere (1940b).

Ertel (1943) treated the nonlinear vorticity equation for a rotating invisid fluid and obtained a solution for stationary motion of harmonic waves of finite lateral extent on a sphere. Craig (1945), Neamtan (1946), Rombakis (1948), Höiland (1951), and Long (1952) have more recently studied the nonlinear vorticity equation in great detail.

In all the previous treatments the fluid has been considered as nonviscous, and motion is independent of height. It is the objective of the present paper to study the fluid motion including the effects of the viscosity and the height dependency of velocity components. On account of frictional effects, the stream functions obtained for the solutions vary with time.

2. Harmonic Wave Solutions of the Nonlinear Vorticity Equation for a (rotating) Viscous Fluid on an Earth whose Variation of Coriolis Parameter with Latitude is Constant

If the vertical velocity is neglected and zero horizontal divergence is assumed, the equations of atmospheric motion and equation of continuity are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right), \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3)$$

where x, y, z are the Cartesian coordinates, positive toward the east, north and upward respectively; u, v the velocity components in the x- and y-direction respectively; $f = 2\Omega \sin\phi$ where Ω is the angular velocity of the rotating earth; ϕ is the latitude; ν is the coefficient of kinematic viscosity, p the pressure and ρ the density.

Equation (3) shows that there exists a stream function ψ such that

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}. \quad (4)$$

If we assume further that fluid is autobarotropic, the vorticity equation of a rotating viscous fluid is

$$\left\{\frac{\partial}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right\} \cdot \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \beta\frac{\partial\psi}{\partial x} = 0 , \tag{5}$$

where

$$\beta = \frac{\partial f}{\partial y} \approx \frac{2\Omega}{a}\cos\phi , \tag{6}$$

a being the radius of the earth.

Assume wave solutions of the form

$$\psi(x,y,z,t) = -Uy + \left\{A\sin(\alpha y + \epsilon) + B\begin{pmatrix}\cos K(x-ct)\cos K\eta y \\ \sin K(x-\eta y-ct)\end{pmatrix}\right\} \cdot g(z)e^{-\xi_1 t} , \qquad \begin{matrix}(7a)\\(7b)\end{matrix}$$

$$\psi(x,y,z,t) = -Uy + \left\{A\sin(\alpha y + \bar{R}z + \epsilon) + B\begin{pmatrix}\cos K(x+Rz-ct)\cos K\eta y \\ \cos K(x-ct)\cos K(\eta y+Rz) \\ \sin K(x-\eta y+Rz-ct)\end{pmatrix}\right\} \cdot e^{-\xi_2 t} , \qquad \begin{matrix}(7c)\\(7d)\\(7e)\end{matrix}$$

where A, B, R, ϵ : arbitrary constants

α, $\bar{R}$, ξ_1, ξ_2, : constants whose values are to be determined

c = wave velocity, positive toward the east

$K = \frac{2\pi}{L}$, L = wave length

In (7a) (7c) (7d) : $\eta = \frac{L}{D}$, D = twice the lateral extent of the waves

In (7b) and (7e) : η = reciprocal of the horizontal slope of the trough and ridge lines

Stream functions (7a), (7c), and (7d) represent waves of finite lateral extent, whereas stream functions (7b) and (7e) represent waves with horizontally tilted troughs and ridges. There is vertical tilt of troughs and ridges in the waves represented by (7c), (7d), and (7e) but no vertical tilt in the waves represented by (7a) and (7b).

It is to be noticed that the mean zonal speed averaged over a complete wave length,

$$\frac{1}{L}\int_0^L u\,dx = U - \alpha A \cos(\alpha y + \epsilon)\, g(z)\, e^{-\xi_1 t}\,, \qquad \text{(8a,b)}$$

$$\frac{1}{L}\int_0^L u\,dx = U - \alpha A \cos(\alpha y + \bar{R} z + \epsilon)\, e^{-\xi_2 t}\,, \qquad \text{(8c,d,e)}$$

varies not only with latitude but also with height and time. Here (8a), (8b), (8c), (8d), and (8e) give the mean zonal speed corresponding to the waves represented by (7a), (7b), (7c), (7d), and (7e), respectively.

Substituting from (7a), (7b) in (5), dividing through with $g(z)e^{-\xi_1 t}$ and arranging terms, we have

$$\begin{aligned} BK\Big\{\Big[K^2(1+\eta^2)(U-c)+\beta\Big] + A\alpha\Big[\alpha^2 - K^2(1+\eta^2)\Big]\cos(\alpha y+\epsilon)\,g(z)\,e^{-\xi_1 t}\Big\}\cdot \\ \cdot\begin{pmatrix}\sin K(x-ct)\cos K\eta y \\ -\cos K(x-\eta y-ct)\end{pmatrix} + \Big\{\alpha^2\Big[\xi_1-\nu\Big(\alpha^2-\frac{g''(z)}{g(z)}\Big)\Big]A\cos(\alpha y+\epsilon) \qquad \begin{matrix}\text{(9a)}\\ \text{(9b)}\end{matrix} \\ + BK^2(1+\eta^2)\Big[\xi_1-\nu\Big(K^2(1+\eta^2)-\frac{g''(z)}{g(z)}\Big)\Big]\begin{pmatrix}\cos K(x-ct)\cos K\eta y \\ \sin K(x-\eta y-ct)\end{pmatrix}\Big\} = 0\,, \end{aligned}$$

where g''(z) denotes the second derivative of g(z). Similarly, substitution of (7c), (7d), and (7e) in (5) gives

$$
\begin{aligned}
&BK\left\{\left[K^2(1+\eta^2)(U-c)+\beta\right]+A\alpha\left[\alpha^2-K^2(1+\eta^2)\right]\cos(\alpha y+\bar{R}z+\epsilon)e^{-\xi_2 t}\right\}\cdot \\
&\cdot\begin{pmatrix}\sin K(x+Rz-ct)\cos K\eta y \\ \cos K(x-ct)\sin K(\eta y+Rz) \\ \cos K(x-\eta y+Rz-ct)\end{pmatrix}+\left\{\alpha^2\left[\xi_2-\nu(\alpha^2+\bar{R}^2)\right]A\sin(\alpha y+\bar{R}z+\epsilon)\right. \\
&\left.+BK^2(1+\eta^2)\left[\xi_2-\nu K^2(1+\eta^2+R^2)\right]\begin{pmatrix}\cos K(x+Rz-ct)\cos K\eta y \\ \cos K(x-ct)\cos K(\eta y+Rz) \\ \sin K(x-\eta y+Rz-ct)\end{pmatrix}\right\}=0\,. \qquad \begin{matrix}(9c)\\(9d)\\(9e)\end{matrix}
\end{aligned}
$$

Thus it is seen that (7a), (7b), (7c), (7d), and (7e) are solutions of (5) providing that

$$
\alpha=\begin{cases}\dfrac{2\pi}{LD}\sqrt{L^2-D^2}\ , & (10a,c,d) \\[2ex] \dfrac{2\pi}{L}\sqrt{1+\eta^2}\ , & (10b,e)\end{cases}
$$

From (9a), (9b), (9c), (9d), and (9e), we have

$$
c=\begin{cases}U-\dfrac{\beta L^2 D^2}{4\pi^2(L^2+D^2)} & (11a,c,d) \\[2ex] U-\dfrac{\beta L^2}{4\pi^2(1+\eta^2)}\ , & (11b,e)\end{cases}
$$

$$
\xi_1=\begin{cases}\nu\left[\dfrac{4\pi^2}{L^2D^2}(L^2+D^2)-\dfrac{g''(z)}{g(z)}\right] & (12a) \\[2ex] \nu\left[\dfrac{4\pi^2}{L^2}(1+\eta^2)-\dfrac{g''(z)}{g(z)}\right]\ , & (12b)\end{cases}
$$

$$\xi_2 = \begin{cases} \nu \dfrac{4\pi^2}{L^2 D^2}\left[L^2 + D^2(1+R^2)\right] & \text{(12c,d)} \\ \nu \dfrac{4\pi^2}{L^2}\left[1+\eta^2+R^2\right] \quad , & \text{(12e)} \end{cases}$$

and

$$\bar{R} = KR \qquad (13)$$

Equations (11a, c, d) give wave velocity in terms of U, β, the wave length L, and the lateral extent D/2, whereas (11b, e) give the wave velocity in terms of U, β, L, and the reciprocal of the horizontal slope of trough and ridge lines of the waves. Since ξ_1, has been assumed as a constant, g(z) must be proportional to the hyperbolic sine and hyperbolic cosine of z, a combination of sine and cosine of z, proportional to z, or independent of z. The last two cases contribute no frictional effect and are of little interest. We shall consider the former two cases which will be shown to be the major cause of the growth of waves under certain circumstances. Let

$$g(z) = G \sinh \zeta z + E \sinh \zeta z \quad , \qquad (14)$$

where G and E are arbitrary constants. It is obvious g(z) becomes a circular function if ζ is imaginary.

Stream functions (7a), (7b), (7c), and (7e) may respectively be written

$$\psi = -Uy + \left\{ A \sin\left(K\sqrt{1+\eta^2}\, y + \epsilon\right) + B \begin{pmatrix} \cos K(x-ct)\cos K\eta y \\ \sin K(x-\eta y - ct) \end{pmatrix} \right\} \cdot \begin{matrix} (15a) \\ (15b) \end{matrix}$$

$$\cdot\, (G \sinh \zeta z + E \cosh \zeta z)\, e^{-\nu\left(K^2(1+\eta^2)-\zeta^2\right)t},$$

$$\psi = -Uy + \left\{ A \sin\left(K\sqrt{1+\eta^2}\, y + KRz + \epsilon\right) \right.$$

$$\left. + B \begin{pmatrix} \cos(x+Rz-ct)\cos K\eta y \\ \cos K(x-ct)\cos K(\eta y + Rz) \\ \sin K(x-\eta y + Rz - ct) \end{pmatrix} \right\} e^{-\nu K^2(1+\eta^2+R^2)t}\,. \quad \begin{matrix} (15c) \\ (15d) \\ (15e) \end{matrix}$$

The first term of the right side of equation (15) gives a constant zonal velocity, the second term a pure zonal current which is a function of latitude, height and time, the last term gives both zonal and meridional velocity which is a function of three coordinates and time. It is of importance to note that waves will always be damped if the vertical profile of velocity components is a sinusoidal function of z, proportional to z, or independent of z. If the vertical profile of the horizontal velocity component is a combination of hyperbolic sine and cosine of z, waves may be damped, unchanged, or amplified, depending respectively on

$$K^2(1+\eta^2) \gtreqless \zeta^2\,, \qquad (16)$$

Examination of (16) reveals that the growth of waves in such a system depends on the balance of the wave length and the lateral extent (or the inclination of trough lines) of the waves with the vertical profile of velocity components. The smaller the horizontal extent of the waves, the greater the damping effect will be. The frictional effect generated by a sinusoidal velocity profile always serves as a damping factor, whereas a hyperbolic sine and cosine profile of velocity serves as an amplifying factor. This can be justified from the following energy considerations. Multiplying (1) (2) by u and v respectively, then adding these two equations and integrating it over the volume of the system, we have

$$\int_V \frac{d}{dt}\left(\frac{u^2+v^2}{2}\right)\rho\, d\tau = -\int_V \left(\frac{dp}{dt} - \frac{\partial p}{\partial t}\right) d\tau + \int_V \nu \left\{ u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\right\} \rho\, d\tau \tag{17}$$

Examining the above equation, one finds that frictional effect tends to decrease the total kinetic energy of the wave system if the velocity components of the system are sine and cosine functions of x, y, and z; whereas it tends to increase the total kinetic energy of the system if the velocity components are hyperbolic sine and cosine functions of x, y and z.[1] The

[1] Friction also tends to increase the kinetic energy of the

velocity components of the solutions with which we are dealing in this section are sine and cosine functions in x and y, but may be independent of z, proportional to z, sine and cosine functions of z, or hyperbolic sine and cosine functions of z. According to the energy considerations it is obvious that waves will always be damped out in the first three cases, since the total kinetic energy of the system will decrease due to frictional effect. However, in the last case the change of total kinetic energy of the system due to frictional effect, and therefore the growth of the wave, depends on the balance of the kinetic energy being fed into the system due to the vertical hyperbolic sine and cosine profile of velocity components, and the kinetic energy being dissipated due to the horizontal sine and cosine profile of the velocity components.

It has been previously pointed out that solutions (15a, b, c, d, e) permit not only vertical but also meridional variations of the mean zonal current. This allows us to approximate a jet stream by a sinusoidal vertical profile, which vanishes at the earth's surface and at $z = H$, say the height of a homogeneous atmosphere, and has a maximum speed at $z = H/2$. For this case stream functions (15a) and (15b)

---

system if u, v are proportional to x^n, y^n, z^n for $n \geq 2$. However, this case is not of interest in the present problem.

become

$$\psi = \left\{A \sin\left(K\sqrt{1+\eta^2}\, y + \epsilon\right) + B\begin{pmatrix}\cos K(x-ct)\cos K\eta y \\ \sin K(x-\eta y - ct)\end{pmatrix}\right\} \cdot \tag{18a, 18b}$$

$$\cdot \left(\sin \frac{\pi}{H} z\right) e^{-\nu\left(K^2(1+\eta^2)+\frac{\pi^2}{H^2}\right)t}$$

It is clear that in such a system frictional effects tend to damp out harmonic waves. Of course atmospheric motion is complicated; wave development is also affected by pressure-density solenoids and vertical motion. Our objective here is to bring out the effect of inertial force and frictional force in the atmospheric motion.

More general solutions of the forms (15a, b, c, d, e) may be constructed and expressed as follows:

$$\psi = -Uy + \left\{A \sin(\alpha y + \epsilon) + \sum_{i=1}^{\infty} B_i \begin{pmatrix}\cos K_i(x-ct)\cos K_i\eta_i y \\ \sin K_i(x-\eta_i y - ct)\end{pmatrix}\right\} \cdot \tag{19a, 19b}$$

$$\cdot \left(G \sinh \zeta z + E \cosh \zeta z\right) e^{-\nu(\alpha^2-\zeta^2)t}$$

$$\psi = -Uy + \sum_{i=1}^{\infty}\left\{A_i \sin(\alpha y + K_i R z + \epsilon) + B_i \begin{pmatrix}\cos K_i(x+Rz-ct)\cos K_i\eta_i y \\ \cos K_i(x-ct)\cos K_i(\eta_i y + Rz) \\ \sin K_i(x-\eta_i y + Rz - ct)\end{pmatrix}\right\} \cdot e^{-\nu K_i^2(1+\eta_i^2+R^2)t}, \tag{19c, 19d, 19e}$$

if

$$K_1^2(1+\eta_1^2)=\alpha^2 . \qquad (20)$$

Equation (20) indicates a connection between the wave length and the lateral extent in (19a), (19c) and (19d), between the wave length and the horizontal inclination of trough-and-ridge-lines in (19b) and (19e). It is interesting to note that in viscous plane flow stationary waves occur if both

$$U=\frac{\beta}{K^2(1+\eta^2)} ,$$

and

$$\zeta^2=K^2(1+\eta^2)$$

are satisfied.

Solutions (19a), (19c) and (19d) contain the solutions of Rossby (1939), Haurwitz (1940a), Craig (1945), and Neamtan (1946) if we put $\nu=\zeta=R=0$ whereas solutions (19b) and (19e) contain the solutions of Machta (1949) and Arakawa (1953) by putting $\nu=\zeta=R=0$.

3. Harmonic Wave Solution of the Nonlinear Vorticity Equation for a Rotating Viscous Fluid on a Sphere

If radial velocity is neglected and zero horizontal divergence is assumed, the two fundamental equations of spherical motion and equation of continuity are respectively reduced to:

$$\left(\frac{\partial}{\partial t}+\frac{U_\theta}{r}\frac{\partial}{\partial\theta}+\frac{U_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\right)U_\theta-2\Omega\cos\theta\,U_\lambda-\frac{U_\lambda^2}{r}\tan\theta$$

$$=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}-\frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{kM}{r}-\frac{1}{2}\Omega^2r^2\sin^2\theta\right) \qquad (21)$$

$$+\nu\left[\nabla^2U_\theta-\frac{U_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\lambda}{\partial\lambda}\right],$$

$$\left(\frac{\partial}{\partial t}+\frac{U_\theta}{r}\frac{\partial}{\partial\theta}+\frac{U_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}\right)U_\lambda+2\Omega\cos\theta\,U_\theta+\frac{U_\theta U_\lambda}{r}\tan\theta$$

$$=-\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\lambda}-\frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\left(-\frac{kM}{r}-\frac{1}{2}\Omega^2r^2\sin^2\theta\right) \qquad (22)$$

$$+\nu\left[\nabla^2U_\lambda-\frac{U_\lambda}{r^2\sin^2\theta}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\theta}{\partial\lambda}\right],$$

$$\frac{\partial}{\partial\theta}(U_\theta\sin\theta)+\frac{\partial U_\lambda}{\partial\lambda}=0\ , \qquad (23)$$

where

$$\nabla^2\equiv\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\ , \qquad (24)$$

is the three dimensional spherical Laplacian operator. θ,λ,r are respectively the colatitude, longitude, and radial distance from the center of the earth; U_θ, U_λ are respectively the velocity components in the direction of the gradient of θ and λ; k and M are the constants of universal attraction and the mass of the earth respectively.

Equation of continuity (23) shows that there exists a stream function Ψ such that

$$U_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\lambda}, \qquad (25)$$

$$U_\lambda = \frac{1}{r}\frac{\partial\Psi}{\partial\theta}. \qquad (26)$$

The radial component of relative vorticity may be written

$$\begin{aligned} q &= \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(U_\lambda\sin\theta) - \frac{\partial U_\theta}{\partial\lambda}\right\} \\ &= \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\right\}\Psi. \end{aligned} \qquad (27)$$

The radial component of absolute vorticity is

$$q_a = 2\Omega\cos\theta + \frac{1}{r^2}\Delta\Psi, \qquad (28)$$

where

$$\Delta \equiv \left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\right\}, \qquad (29)$$

is the spherical surface-Laplacian operator.

If we further assume that the fluid is homogeneous, the equations of motion may further be written as follows:

$$\frac{\partial U_\theta}{\partial t} - U_\lambda(q + 2\Omega\cos\theta)$$

$$= -\frac{1}{r}\frac{\partial}{\partial\theta}\left[(U_\theta^2 + U_\lambda^2) + \frac{p}{\rho} - \frac{kM}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta\right] + \nu R_\theta\,, \tag{30}$$

$$\frac{\partial U_\lambda}{\partial t} + U_\theta(q + 2\Omega\cos\theta)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}\left[(U_\theta^2 + U_\lambda^2) + \frac{p}{\rho} - \frac{kM}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta\right] + \nu R_\lambda\,, \tag{31}$$

where

$$R_\theta = \nabla^2 U_\theta - \frac{U_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\lambda}{\partial\lambda}\,, \tag{32}$$

$$R_\lambda = \nabla^2 U_\lambda - \frac{U_\lambda}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\theta}{\partial\lambda}\,. \tag{33}$$

The vorticity equation of a rotating viscous fluid on a sphere may be written

$$\frac{\partial}{\partial t}\Delta\Psi + \frac{1}{r^2\sin\theta}\left(\frac{\partial\Psi}{\partial\theta}\frac{\partial}{\partial\lambda} - \frac{\partial\Psi}{\partial\lambda}\frac{\partial}{\partial\theta}\right)(\Delta\Psi + 2\Omega r^2\cos\theta)$$

$$= \frac{\nu}{r^2}\left[\Delta^2\Psi + r\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\left(\frac{\Delta\Psi}{r}\right)\right]. \tag{34}$$

To solve (34) we assume a wave solution of the form

$$\Psi(\theta,\lambda,r,t) = -\omega r^2\cos\theta + Y_n(\theta,\lambda,t)f(r)e^{-\nu st}\,, \tag{35}$$

where ω is a constant, $Y_n(\theta,\lambda,t)$ an associated Legendre function of order n with time dependency, s is a parameter which will be shown to be related to f(r). The term, $-\omega r^2\cos\theta$, represents the part of the stream function resulting from a solid rotation of angular velocity ω relative to the rotating earth. This term is independent of time due to the fact that solid rotation is not affected by friction. By making use of the fact that Y_n satisfies the equation

$$\Delta Y_n + n(n+1)Y_n = 0 \,, \tag{36}$$

and also that:

$$\Delta \cos\theta = -2\cos\theta \,, \tag{37}$$

we have from (35):

$$\Delta\Psi + 2\Omega r^2 \cos\theta = 2(\Omega+\omega) r^2 \cos\theta - n(n+1)Y_n f(r) e^{-\nu s t} \tag{38}$$

After substitution of (35) and (38) into (33), the vorticity equation reduces to

$$\frac{\partial Y_n}{\partial t} + \left\{\omega - \frac{2(\Omega+\omega)}{n(n+1)}\right\}\frac{\partial Y_n}{\partial \lambda} - \nu\left\{s - \left[\frac{n(n+1)}{r^2} - \frac{f''(r)}{f(r)}\right]\right\} Y_n = 0 \,, \tag{39}$$

It is evident that the associated Legendre function $Y_n(\theta,\lambda,t)$ is a solution of differential equation (39) if

$$Y_n(\theta,\lambda,t) = Y_n(\theta,\lambda - \Omega_n t) \,, \tag{40}$$

and

$$\frac{d^2 f}{d r^2} + \left[s - \frac{n(n+1)}{r^2}\right] f = 0 \,, \tag{41}$$

where

$$\Omega_n = \omega - \frac{2(\Omega+\omega)}{n(n+1)} \,. \tag{42}$$

Differential equation (41) has a solution

$$f(r) = r^{\frac{1}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r) \ , \qquad (43)$$

where $Z_{n+\frac{1}{2}}$ is a Bessel function of order $n + \frac{1}{2}$.

Solution (35) is, therefore

$$\Psi(\theta,\lambda,r,t) = -\omega r^2 \cos\theta + \left\{A_n P_n(\cos\theta) + \sum_{m=1}^{n} A_n^m P_n^m(\cos\theta)\cos m(\lambda - \Omega_n t + B_n^m)\right\} \cdot$$
$$\cdot\, r^{\frac{1}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r) e^{-\nu s t} . \qquad (44)$$

Where A_n and A_n^m ($m = 1,2, \ldots , n$) are $n + 1$ amplitude factors, B_n^m ($m = 1,2, \ldots , n$) are n phase angles. Whether the harmonic waves of this system are amplified, unchanged, or damped, depends respectively on

$$S \lesseqgtr 0 , \qquad (45)$$

where s is associated with the argument of Bessel function $Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r)$. For large values of $s^{\frac{1}{2}} r$ (in the atmosphere $r = a + z$ and $s^{\frac{1}{2}} a$ is of the order of 10^3) [2]; $r^{\frac{1}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r)$ behaves like a circular function or an exponential function according to whether s is a positive or negative number. This is similar to what we have found for the case of viscous flow

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[2] $a \sim 6.37 \times 10^6$ m, $s^{\frac{1}{2}} \sim \frac{\pi}{H} \sim 3 \times 10^4\ m^{-1}$.

on a plane. It is also interesting to note that $s = \frac{n(n+1)}{(a+Z)^2} \approx 0$ if $f(r)$ is proportional to or independent of r. For this case flow is approximately equivalent to non-viscous flow.

If all amplitude factors A_n^m vanish except for a certain m, (45) becomes

$$\Psi = -\omega r^2 \cos\theta + \left\{A_n P_n(\cos\theta) + A_n^m P_n^m(\cos\theta)\cos m(\lambda - \Omega_n t + B_n^m)\right\} r^{\frac{1}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r) e^{-\nu s t}, \tag{46}$$

for m = 1,2, ... , n, which of course is a solution of (34). It is seen from (46) and (42) that if the relative angular velocity, ω, is absent, the westward wave velocity increases with decreasing wave length and increasing lateral extent of the waves. The corresponding angular velocity of the current in the direction of the gradient of θ and λ are, respectively,

$$\frac{d\theta}{dt} = -\frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\lambda} = m\cos\theta\left\{A_n^m P_n^m(\cos\theta)\sin m(\lambda - \Omega_n t + B_n^m)\right\} r^{-\frac{3}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r) e^{-\nu s t}, \tag{47}$$

$$\frac{d\lambda}{dt} = \frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta} = \omega - \left\{A_n \frac{dP_n(\cos\theta)}{d(\cos\theta)} + A_n^m \frac{dP_n^m(\cos\theta)}{d(\cos\theta)}\cos m(\lambda - \Omega_n t + B_n^m)\right\} \cdot r^{-\frac{3}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r) e^{-\nu s t}. \tag{48}$$

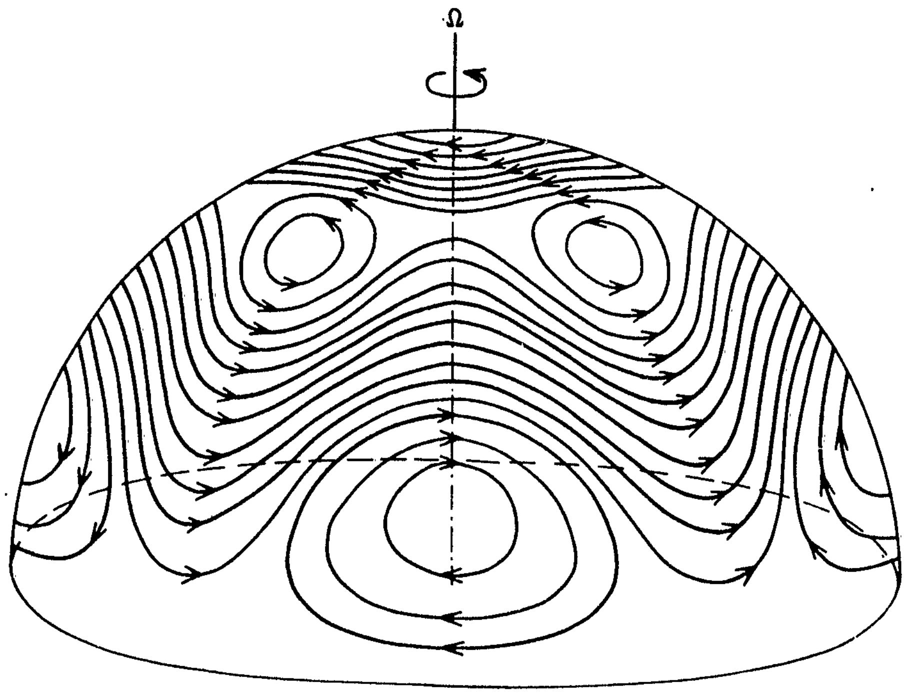

Fig. 1

Comparison of the corresponding terms in (46), (47), and (48) shows that the first term of the right hand side of (46) is the part of the stream function resulting from a solid rotation of angular velocity ω relative to the rotating earth. The second term of the right hand side of (46) gives a purely zonal current, whereas the last term gives both zonal and meridional flows.

The part of the stream function representing the last term of (46) gives 2m nodal meridians, or m waves along each latitudinal circle. In addition:

$$P_n^m(\cos\theta) = \sin^m\theta \, \frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m} \ .$$

This is a polynomial in cos θ of degree (n - m) which vanishes on n - m parallels between the poles. Therefore, this part of the stream function gives cells over the globe which is total 2m (n - m + 1) in number. These traveling cells possess a period

$$T_n^m = \frac{2\pi}{m\left[\omega - \frac{2(\Omega+\omega)}{n(n+1)}\right]} \ . \tag{49}$$

It is interesting to note that invisid, spherical-flow waves become stationary if

$$\omega = -\frac{2\Omega}{(n-1)(n+2)} \ . \tag{50}$$

It is seen from (46) that no stationary wave is possible in viscous flow over a rotating sphere, since this condition would also require $s = 0$. But $Z_{n+\frac{1}{2}}(0) = 0$ for $n \neq -1/2$. However, quasi-stationary waves, which may be defined as waves with zero wave-velocity but changing wave-amplitude, may occur when condition (50) is satisfied. In order to illustrate the distribution of harmonic waves over a spherical surface, the stream function of an example for quasi-stationary case with $m = 4$ and $n = 5$ has been sketched as shown in fig. 1.

4. Conclusions

It has been shown that solutions of the nonlinear vorticity equation for a rotating viscous fluid yield waves which vary with both height and time. Whether these waves are damped, neutral, or amplified depends on the longitudinal and latitudinal extents of the waves, and also on the vertical profile of their stream function (therefore their velocity components). This is evidently due to the effect of Navier-Stokes friction.

Plane-flow waves, for which velocity components are proportional to or independent of height or have a sinusoidal vertical profile, appear always to be damped; on the other hand, waves, for which velocity components vary with the hyperbolic sine and cosine of z, may be amplified, neutral, or damped, depending respectively on $K^2(1+\eta^2) \lesseqqgtr \zeta^2$. The velocity of propagation of these waves is

$$C = U - \frac{\beta}{K^2(1+\eta^2)} ,$$

which is independent of height. However, it is of importance to note that here U is not generally equal to the mean zonal velocity.

For spherical flow, waves are amplified, neutral or damped, depending respectively on $s \lesseqqgtr 0$, where s is associated with the vertical profile of the stream function, which was found to be proportional to $r^{\frac{1}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r)$. For large values of $s^{\frac{1}{2}} r$, $r^{\frac{1}{2}} Z_{n+\frac{1}{2}}(s^{\frac{1}{2}} r)$ behaves like a sinusoidal or exponential function of $s^{\frac{1}{2}} r$, depending respectively on whether s is a positive or negative quantity. The period of these spherical waves was found to be

$$T_n^m = \frac{2\pi}{m\left[\omega - \frac{2(\Omega+\omega)}{n(n+1)}\right]} .$$

It was also found that no stationary wave is possible in spherical viscous flow, although quasi-stationary waves, which may be defined as waves with zero wave-velocity but changing wave-amplitude, may occur when

$$\omega = -\frac{2\Omega}{(n-1)(n+2)} .$$

Harmonic waves are generally damped in a laminar viscous flow between two rotating plates or spherical shells, since waves must vanish on the solid boundaries. This permits only a sinusoidal vertical profile of the stream function. It is seen from the properties of Bessel function that for a certain integer n the greater the value of s, the smaller the distance between neighboring zeros of $Z_{n+\frac{1}{2}}(s^{\frac{1}{2}}r)$. This means that the smaller the distance between two neighboring zeros of the vertical profile of the stream function, the greater is the damping factor.

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